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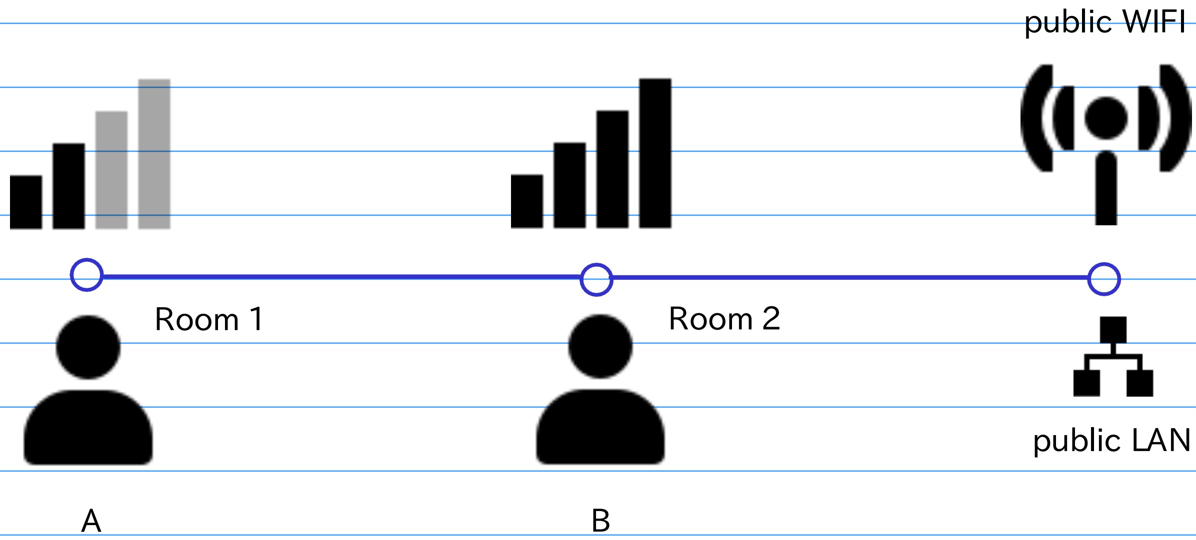
Dormitory WIFI game (original)



Students are thinking if they shall set up their personal WIFI routers which may increase radio interference and slow down access speed of all.

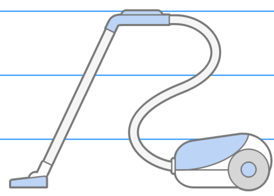
payoff matrix

		B: cooperate	B: defect
A	cooperate	3, 7	1, 10
	defect	7, 5	5, 8

Dormitory cleaning game (contributed by W)

Payoff of a player i (i=1, 2, ..., N)

	at least one other people cleans	all others defect
clean	0	-1
defect	1	-5



*Clean also means to clean the trash pack.

The TTC algorithm

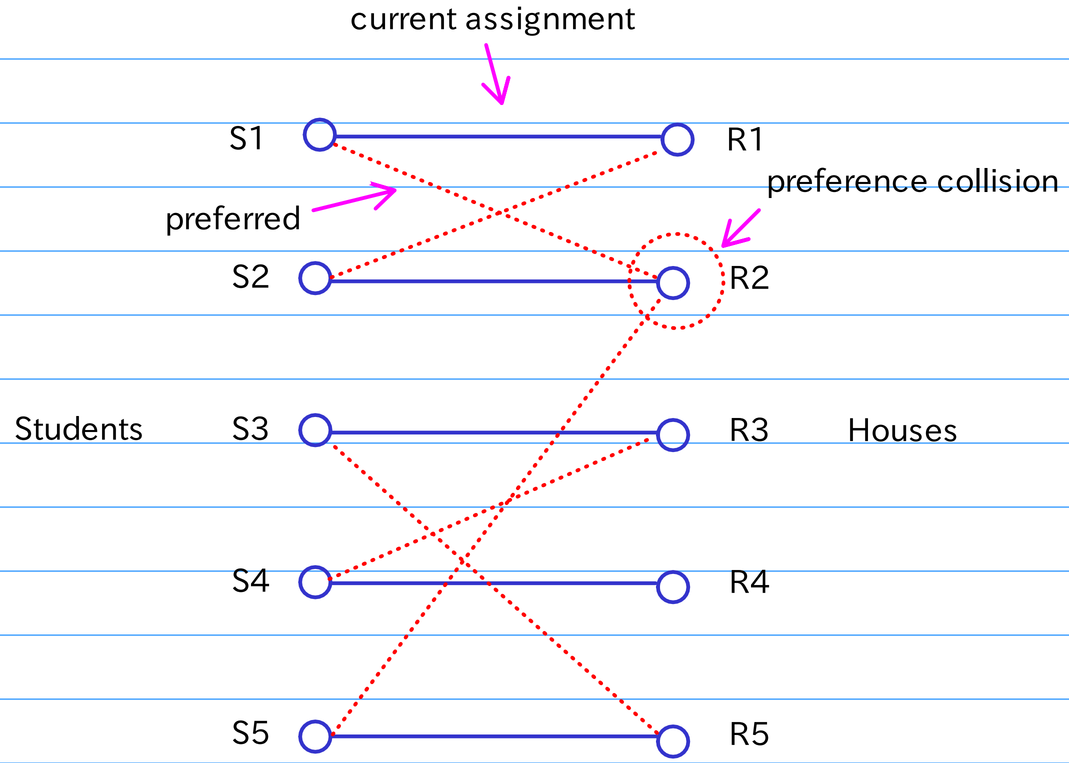
(An application of graph theory and game theory)

House allocation problem (also occurs in trading)

Suppose there are n students assigned to n single houses but some are not satisfied with the assignment, i.e., they have different preference of houses.

Q: How to find an "optimal" re-assignment?

Notice there may exist preference collision, thus we need to a "good" mechanism.



Method 0: full dictatorship (e.g., Shishu-Kan's domitory)

Assgin without considering students' preference (e.g., random assignment).

=> Fair but the satisfaction is usually not good (not Pareto efficient. E.g., S1 and S2).

Method 1: serial dictatorship (e.g., Shishu-Kan's commons)

Decide an order and (in that order) let students choose houses one by one.

(Note: The order can be decided randomly, by seniority, or other priority.)

Theorem 1. Serial dictatorship is Pareto-efficent and strategy-proof.

Pareto-efficient: no player can get a better state without making some other worse.

Strategy-proof: revealing preferences truthfully is a dominant strategy.

Proof.

Pareto-efficiency: At any time a student can choose the best free house.

Strategy-proof: obvious. ∎

Individually rational: each student gets a house that is at least as good as the initial one.

Clearly Serial dictatorship cannot guarantee individually rationality.

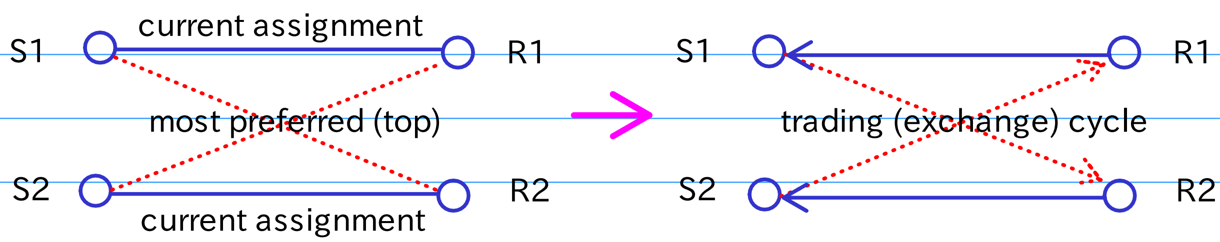
Let us introduce a new algorithm.

2012 Nobel Prize in Economics "for the theory of stable allocations and the practice of market design."

Top Trading Cycle (TTC) Algorithm

Developed by David Gale'62 and published by Hilbert Scarf & Lloyd Shapley'74.

Observation (top trading cycle)



set all students and all houses "unfixed"

while there exists unfixed students

V = {unfixed students}, H = {unfixed houses}

w(i) = current assigned house of student i, i ¥in V

v(i) = house of the top (i.e., most preferred) free house of student i, i ¥in V

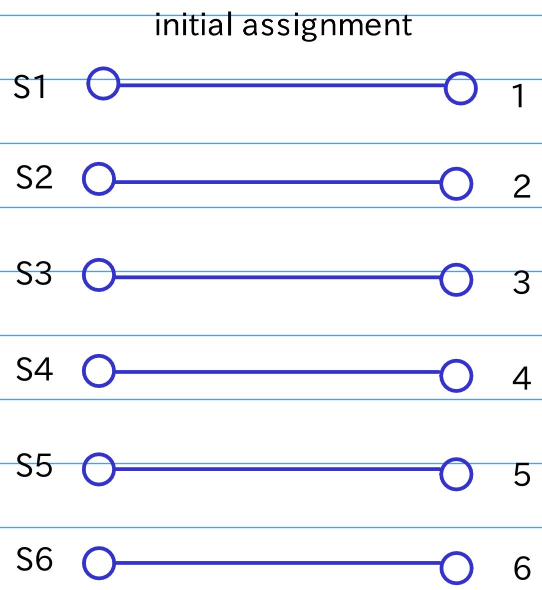
Let G = (V, E) be a digraph with E = {(i, v(i) | i ¥in V)} ¥cup {(w(i), i) | i ¥in V}.

Find a (directed) cycle C in G and following that cycle fix students and houses.

TTC algorithm exercise

Decide your preference (randomly).

E.g., 3, 5, 1, 6, 2, 4



S1:

S2:

S3:

S4:

S5:

S6:

Two questions on the TTC algorithm:

1. Feasibility: Why there can always find a cycle in G?

2. Running time.

Theorem 2. TTC is Pareto-efficent, strategy-proof and Individually rational.

(Gale, Scarf, and Shapley'74)

Theorem 3. TTC is the only algorithm that is Pareto-efficent, strategy-proof and Individually rational. (Ma'94)

TTC algorithm in practice

In 2012, the New Orleans Recovery School District adopted a school version of TTC.

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Graph: tree

Tree: a connected (undirected) graph with no cycle.

Lemma 1: $T = (V, E)$ is a tree => $|E| = |V| - 1$.

Proof. By mathematical induction.

Denote $|V|$ and $|E|$ by n and m, respectively.

Step 1: $n = 1$ => T can be just a single node (with no edge) => OK

Step 2: Assume the lemma is true for any tree with $n - 1$ nodes.

Step 3: Consider a tree $T = (V, E)$ of $n >= 2$ nodes.

Since $n >= 2$, T is connected and has no cycle, there must exist a node v with degree 1.

Remove that v and the only edge incident to v from T, we get a tree T'.

Notice T' has $n-1$ nodes and $m-1$ edges.

Apply the assumption in Step 2 to T', we have $(m-1) = (n-1) - 1$. => $m = n - 1$. ∎

Appendix: mathematical induction

Suppose we want to prove some proposition.

Step 1: Prove it is true for the case of $n = 1$.

Step 2: Assume it is true for all cases of $n - 1$ (or any $k < n$).

Step 3: Prove it for the case of n.

=> It is true for all natual numbers.

Bad (Wrong) use of mathematical induction.

Suppose we want to prove Lemma 1.

Why is it wrong?

Step 1: For $n = 1$, it is true. (OK)

Step 2: Assume it is true for all trees T' with $n - 1$ nodes. (OK)

Step 3: Appending a node v by an edge to T' gives us a tree T of n nodes.

Since the number of nodes and the number of edges increase by one respectively, the lemma has been proved.