* advanced

mini report

- Consider the following LP problem (P) $\min c^T x$ s.t. $Ax \geq b$, where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, x is the decision variable. Show the dual problem of (P), then describe and prove the weak duality theorem.

- ① First transform (P) to a standard formulation.

Let $x = x_1 - x_2$, $x_1, x_2 \in \mathbb{R}^n$ and $x_1, x_2 \geq 0$.

$$(P) \Rightarrow (P') \quad \min c'^T x' \quad \text{s.t. } A'x' \geq b,\ x' \geq 0$$

where $c' = \begin{pmatrix} c \\ -c \end{pmatrix} \in \mathbb{R}^{2n}$, $x' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n}$, $A' = (A \ \ -A) \in \mathbb{R}^{m \times 2n}$

$$\Rightarrow (D') \quad \max b^T y \quad \text{s.t. } A'^T y \leq c',\ y \geq 0$$

$$\Leftrightarrow (D') \quad \max b^T y \quad \text{s.t. } \begin{pmatrix} A^T \\ -A^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -c \end{pmatrix},\ y \geq 0$$

$$\Leftrightarrow (D) \quad \max b^T y \quad \text{s.t. } A^T y = c,\ y \geq 0$$

② $\forall$ feasible x of (P) and feasible y of (D),

$$b^T y \leq c^T x$$

③ Proof:

$$b^T y = y^T b \overset{\substack{y \geq 0 \\ b \leq Ax}}{\leq} y^T (Ax) = (y^T A) x = x^T (A^T y)^T$$

$$\overset{A^T y = c}{=} x^T c = c^T x \quad \blacksquare$$

Mini Report #8 (melon problem)

A simple method using two formulations (w/o discount)

xij: the number of type-i cases that pack exactly j melon(s)

```
var x11 integer, >=0;
var x12 integer, >=0;
var x21 integer, >=0;
var x22 integer, >=0;
var x23 integer, >=0;
var x24 integer, >=0;
var x31 integer, >=0;
var x32 integer, >=0;
var x33 integer, >=0;
var x34 integer, >=0;
var x35 integer, >=0;
var x36 integer, >=0;


# find an optimal solution if there is no discount.
# minimize cost: 1100 * x11 + 1100 * x12 + 1100 * x21 + ...

# find an optimal solution if there is a discount
minimize cost: 880 * x11 + 880 * x12 + 880 * x21 + ...

# constraints

# Comment out the next line to for no discount.
# subject to c1: x11 + x12 + x21 + ... <= 3
subject to c1: x11 + x12 + x21 + ... >= 4

# total number of melons = 14
subject to c2: x11 + 2 * x12 + x21 + 2 * x22 + ... = 14

end;
```

Mini Report #8 (melon problem) - advanced

A solution using one formulation

Q1: how to set the discount indicator?
Q2: how to decide the discount value?

Standard trick: M-method (big em method)

```
var ...
var discount binary; # 1 if and only # cases >= 4
var y; # 0 if discount = 0; otherwise the discount value

param M := 10000; # Choose a large enough number

# objective function
minimize cost: 1100 * x11 + ... 2100 * x36 - y

# constraints

# total number
subject to c1: x11 + 2 * x12 + x21 + ... + 6 * x36 = 14

# discount should be 0 if and only if # cases >=4
subject to c2: x11 + ... + x36 - 3 <= M * discount;
subject to c3: x11 + ... + x36 - 4 >= M * (discount -1);

# y is at most 20% of the normal price when discount=1
subject to c4: y <= 0.2 * (1100 * x11 + ... + 2100 * x36);
subject to c5: y <= M * discount;

end;
```

Discussion (including other methods).

branch-and-bound method

Ex. 0-1 knapsack problem

- branch : choose some x_k, branch according to $x_k = 0, 1$
- bound : discard a subproblem if it is impossible to get a better solution (than the current best)

usually by testing a relaxation problem

(IP) $\xrightarrow{\text{relaxation}}$ (LP)

$$\max \sum_{i=1}^{n} c_i x_i \qquad\qquad \max \sum c_i x_i$$

$$\text{s.t.} \sum_{i=1}^{n} a_i x_i \le b \qquad\qquad \text{s.t.} \sum a_i x_i \le b$$

$$x_i \in \{0,1\},\ i = 1, \cdots, n \qquad\qquad 0 \le x_i \le 1,\ \forall i$$

- optimal value of (IP) $\le$ optimal value of (LP)
- It is easy to calculate the optimal solution of (LP)

Assume $\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \cdots \ge \frac{c_n}{a_n}$

$$\Rightarrow x_i^* = \begin{cases} 1 & 1 \le i \le i^* = \max\limits_{0 \le j \le n} \left\{ j \,\middle|\, \sum_{k=1}^{j} a_k \le b \right\} \\ \left(b - \sum_{j=1}^{i^*} a_j\right) / a_{i^*+1}, & i = i^*+1 \quad (\text{if } i^* < n) \\ 0 & i^*+1 < i \le n \end{cases}$$

UB: upper bound, CB = current best

$Z = 3x_1 + 4x_2 + x_3 + 2x_4$

s.t. $2x_1 + 3x_2 + x_3 + 3x_4 \le 4$

$\cdots \; UB = \lfloor Z(1, \frac{2}{3}, 0, 0) \rfloor = 5$

$CB = Z(1, 0, 1, 0) = 4$

$x_2 = 0$

$Z = 3x_1 + x_3 + 2x_4$

s.t. $2x_1 + x_3 + 3x_4 \le 4$

$UB = \lfloor Z(1, 0, 1, 1/3) \rfloor = 4$

×

(no better solution)

$x_2 = 1$

$UB = \lfloor Z(1/2, 1, 0, 0) \rfloor = 5$

$Z = 4 + 3x_1 + x_3 + 2x_4$

s.t. $2x_1 + x_3 + 3x_4 \le 1$

$x_1 = 0$

$Z = 4 + x_3 + 2x_4$

s.t. $x_3 + 3x_4 \le 1$

$UB = \lfloor Z(0, 1, 1, 0) \rfloor = 5$

and integer solution

O

(finished)

↑

Optimal solution

$x_1 = 1$

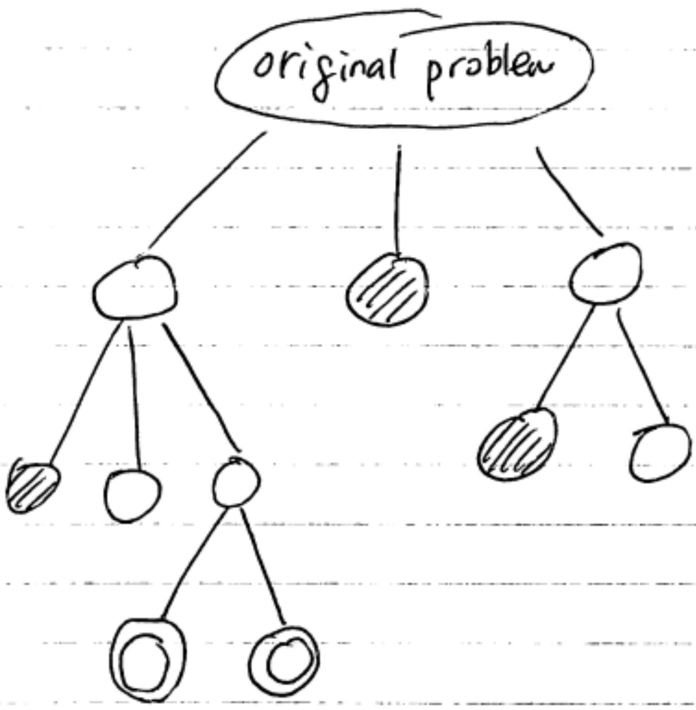
$Z = 7 + x_3 + 2x_4$

s.t. $x_3 + 3x_4 \le -1$

× (not feasible)

BnB in general

- branch : branch the original problem into several subproblems such that the optimal solution can be obtained by solving all the subproblems.
- bound : discard a subproblem if we cannot get a better solution from it than the current best solution.



active problem : unfinished problem

$\{\text{active problem}\} = \phi \Rightarrow$ done